Non-stochasticity of time-dependent quadratic Hamiltonians and the spectra of canonical transformations

1986 J. Phys. A: Math. Gen. 19 521

(http://iopscience.iop.org/0305-4470/19/4/013)

J. Phys. A: Math. Gen. 19 (1986) 521-531. Printed in Great Britain

Non-stochasticity of time-dependent quadratic Hamiltonians and the spectra of canonical transformations

George A Hagedorn†, Michael Loss‡ and Joseph Slawny§

† Department of Mathematics, Virginia Polytechnic Institute and State University, Blacksburg, Virginia 24061, USA

‡ Department of Physics, Princeton University, Princeton, New Jersey 08544, USA

§ Department of Physics, Virginia Polytechnic Institute and State University, Blacksburg, Virginia 24061, USA

Received 20 December 1984, in final form 19 July 1985

Abstract. We study the quantum mechanical evolution generated by Hamiltonians which are quadratic polynomials in q and p with time-dependent coefficients. This evolution is determined by a unitary implementation of the phase flow of the corresponding classical Hamiltonian. In the case of quadratic Hamiltonians which are periodic in time, the Floquet operator is shown to have either a pure point spectrum or a purely transient absolutely continuous spectrum. Thus, the motion is non-stochastic. In a simple model of a quadratic Hamiltonian with random time dependence, the quantum mechanical motion is shown to be non-stochastic almost surely.

1. Introduction

In this paper we study the evolution generated by quantum mechanical Hamiltonians which are self-adjoint quadratic polynomials in q and p with time-dependent coefficients. We show that this evolution is determined, up to a phase factor, by an explicit unitary implementation of the corresponding classical phase flow. This allows us to analyse the large-time asymptotics of the quantum mechanical motion in some specific cases by studying the associated classical motion. The specific cases we study are time-periodic quadratic Hamiltonians and a class of quadratic Hamiltonians with random time dependence. The quantum evolutions for these systems are (almost surely in the random case) non-stochastic in the sense that their long-time behaviours are well behaved.

To make this last statement precise, let us first consider the case of time-periodic quadratic Hamiltonians. Let $H(t)$ be a quadratic polynomial in q and p for each t, and assume the coefficients of these polynomials have piecewise continuous time dependence which is periodic with period T. Under these circumstances, there exists a unitary propagator $U(s, t)$ with the property that, if $\Psi(t)$ is a solution to the Schrödinger equation $\mathrm{i}\,\partial\Psi/\partial t = H(t)\Psi$, then $\Psi(s) = U(s, t)\Psi(t)$. We define the Floquet operator of the system to be $U(T, 0)$. The significance of this operator is due to the fact that its spectral properties reflect the qualitative behaviour of the motion generated by $H(\cdot)$. The analogues of bound states for time-periodic Hamiltonians are the eigenvectors of the Floquet operator. At times $T, 2T, 3T, \ldots$, such a state is equal to the state at time 0 multiplied by a phase factor. At time nT, the phase factor is simply the nth power of the corresponding eigenvalue of $U(T, 0)$. One should note that if $H(t)$ is independent of time, then eigenfunctions of $H(t) = H(0)$ are also eigenfunctions of $U(t, 0) = \exp(-\mathrm{i}tH(0))$ for any t. Similarly, the analogue of the absolutely continuous spectrum for a time-independent Hamiltonian is the absolutely continuous spectrum of the Floquet operator for a time-periodic system. If one further refines the absolutely continuous spectrum, as has been done by Avron and Simon (1981), then the transient absolutely continuous spectrum (Avron and Simon 1981) of the Floquet operator corresponds to the 'non-stochastic' part of the absolutely continuous spectrum. In most physical examples, states in the absolutely continuous subspace correspond to the particle moving off to infinity in phase space as time goes to $\pm\infty$. The transient absolutely continuous spectrum was defined in Avron and Simon (1981) to characterise states which move to infinity in an ordinary, non-chaotic way. (For a good discussion of the connection between spectral theory and dynamical behaviour, see Avron and Simon (1981), Perry (1984, ch 1).)

Our results concerning time-periodic quadratic Hamiltonians are summarised by the following theorem.

Theorem 1. Let $H(t)$ be quadratic in q and p and periodic in t with period T. Assume that the time dependence of the coefficients is piecewise continuous. Then the Floquet operator $U(T, 0)$ has either a strictly pure point spectrum, or has a strictly transient absolutely continuous spectrum.

We next wish to consider some systems which have a simple random time dependence. For simplicity, we will consider a specific example, although one could very easily deal with more general cases.

Let $H(t) = p^2/2 + \lambda(t)x^2$, in one dimension, where $\lambda(t)$ is chosen to be a constant, ω_k, for $k \leq t < k+1$. The ω_k will be chosen according to a probability distribution μ. We assume that this distribution is non-trivial in the following sense. For each ω in the support of μ, consider the classical Hamiltonian $H_\omega = p^2/2 + \omega x^2$. In the next section, we will show that there exists a unique $M_\omega \in \mathrm{SL}(2, \mathbb{R})$, such that the dynamics governed by H_ω have

$$\begin{pmatrix} q(1) \\ p(1) \end{pmatrix} = M_\omega \begin{pmatrix} q(0) \\ p(0) \end{pmatrix}.$$

Let G be the subgroup of $\mathrm{SL}(2, \mathbb{R})$ that is generated by all such M_ω, for ω in the support of μ. Our non-triviality assumption is that G must contain at least two elements of $\mathrm{SL}(2, \mathbb{R})$ with no common eigenvector.

In contrast to the periodic case studied above, there is no obvious operator for this model whose spectral properties determine the long-time behaviour of the quantum motion. That is, there is no obvious analogue of the Floquet operator. Therefore, we must use some other notion of what it means to be 'non-stochastic'. What we will prove is the following theorem.

Theorem 2. Let $H(t)$ be chosen as described above, with the non-triviality condition satisfied. Let $\Psi(t)$ be the solution to the Schrödinger equation determined by $H(t)$. If $\Psi(0) \in \mathscr{S}$ (Schwartz space), then for almost all choices of the sequence ω_k, $|\langle\Psi(0), \Psi(t)\rangle|$ decays faster than any inverse power of t as $t \to \infty$.

This theorem shows that the motion asymptotically obeys the conditions that define the transient absolutely continuous subspaces for a time-independent Hamiltonian (Avron and Simon 1981). Ψ belongs to the transient absolutely continuous subspace for H if, and only if, Ψ is a limit of vectors φ which have the property that $|\langle\varphi, \exp(-itH)\varphi\rangle|$ decays faster than any inverse power of t. For this reason, we interpret the conclusion of theorem 2 to mean that the motion is almost surely non-stochastic as $t \to \infty$.

For convenience, we will concentrate on the analysis of the one-dimensional case and simply comment about the generalisations to n dimensions.

In the next section, we discuss the relationship between the classical and quantum motions associated with quadratic Hamiltonians. In § 3, we study the time-periodic case and prove theorem 1 in the one-dimensional case. We prove theorem 2 in § 4.

2. The connection between classical and quantum motions

In this section we will show that in the case of quadratic Hamiltonians, classical motion determines quantum motion modulo phase factors. One could compute the phases without much difficulty, but to do so would only complicate matters. As mentioned above, we will restrict attention to the one-dimensional case and comment about the generalisation to n dimensions. Some of this material must be 'folk wisdom', but we are not aware of specific references (see note added in proof).

We will concentrate on Hamiltonians of the form

$$H(t) = \alpha(t)p^2 + \beta(t)[p\cdot q + q\cdot p] + \gamma(t)q^2 + \delta(t)p + \varepsilon(t)q + \zeta(t),$$

where the coefficients $\alpha(t)$, $\beta(t)$, $\gamma(t)$, $\delta(t)$, $\varepsilon(t)$ and $\zeta(t)$ are real and piecewise continuous. By a slight abuse of notation, we will let $H(t)$ denote both a classical and a quantum Hamiltonian. We will assume that $\zeta(t) = 0$ since the term $\zeta(t)$ in $H(t)$ does not affect the classical motion, and simply gives rise to the trivial phase factor $\exp(-i\int_t^s \zeta(r)\,dr)$ in the quantum propagator $U(s, t)$.

There are several ways to see that the classical motion determines the quantum motion in the case of quadratic Hamiltonians. For example, one can use the functions of Hagedorn (1981, 1985) to reduce the propagation of quantum states to the solution of Hamilton's equations. However, the most illuminating way is to compare the Lie algebras of classical Poisson brackets and quantum commutators.

In classical mechanics (Arnold 1978, Goldstein 1965), one defines the Poisson bracket of two functions F and G on phase space by

$$\{F, G\} = \sum_{j=1}^{n} \frac{\partial G}{\partial p_i}\frac{\partial F}{\partial q_i} - \frac{\partial G}{\partial q_i}\frac{\partial F}{\partial p_i}.$$

The real polynomials in p and q of degree less than or equal to two form a real Lie algebra under the operation $\{\cdot, \cdot\}$, and Hamilton's equations can be written (in one space dimension) as the system

$$dq/dt = \{q, H(t)\}$$

$$dp/dt = \{p, H(t)\}.$$

In quantum mechanics, the analogous operation is the commutator (up to a factor of $-i$). The self-adjoint operators corresponding to real polynomials of degree less than or equal to two form a Lie algebra under the operation

$$(A, B) = -i[A, B] = -iAB + iBA.$$

Also, given such an operator A, if we define $A(t) = U(t,0)\,A\,U(0,t)$, then formally we have

$$\mathrm{d}A(t)/\mathrm{d}t = -\mathrm{i}[A(t), H(t)].$$

In particular, if we choose A to be q and p, we obtain analogues of Hamilton's equations. For A equal to q or p, one can make these formal computations rigorous by interpreting them in the following weak sense. Let φ and Ψ be elements of the domain of the usual harmonic oscillator Hamiltonian. We then interpret the above equation to mean

$$(\mathrm{d}/\mathrm{d}t)\langle\varphi, A(t)\Psi\rangle = -\mathrm{i}\langle A(t)^*\varphi, H(t)\Psi\rangle + \mathrm{i}\langle H(t)^*\varphi, A(t)\Psi\rangle.$$

For A equal to q or p, it is not difficult to show that all the quantities make sense and that the equation is satisfied.

The correspondence between the classical and quantum motions in the case that we are considering is the following: the two Lie algebras are isomorphic, and consequently, the motions of $q(t)$ and $p(t)$ are the same. The latter fact is the remarkable one. For any piecewise continuous quadratic Hamiltonian $H(t)$, the classical and quantum motions for $q(t)$ and $p(t)$ are given by

$$\begin{pmatrix} q(t) \\ p(t) \end{pmatrix} = M(t)\begin{pmatrix} q \\ p \end{pmatrix} + \begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix}, \tag{1}$$

where $M(t)$ is a matrix.

Due to the irreducibility of the Schrödinger representation of the Lie algebra of q and p in quantum mechanics, the quantum version of equation (1) determines the unitary propagator $U(s,t)$ up to a time-dependent phase. Thus, up to phases, the classical flow determines the quantum evolution.

For future reference, let us make several observations. First, note that the matrix $M(t)$ in equation (1) must belong to $\mathrm{SL}(2,\mathbb{R})$. We may see this by explicit computation, or by using the fact that the classical phase flow preserves volume. Furthermore, explicit examples show that every matrix in $\mathrm{SL}(2,\mathbb{R})$ may occur. Second, let $T\left[\begin{pmatrix} \alpha \\ \beta \end{pmatrix}\right]$ be the translation of phase space given by the relation

$$T\left[\begin{pmatrix} \alpha \\ \beta \end{pmatrix}\right]\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} q+\alpha \\ p+\beta \end{pmatrix}.$$

Then equation (1) may be written as the composition

$$\begin{pmatrix} q(t) \\ p(t) \end{pmatrix} = T\left[\begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix}\right] M(t)\begin{pmatrix} q \\ p \end{pmatrix}. \tag{2}$$

For any linear operator $M \in \mathrm{GL}(2,\mathbb{R})$, we have the relation

$$MT\left[\begin{pmatrix} \alpha \\ \beta \end{pmatrix}\right] M^{-1} = T\left[M\begin{pmatrix} \alpha \\ \beta \end{pmatrix}\right]. \tag{3}$$

Thus, the affine actions given by equations (1) and (2) are representations of the semi-direct product of $\mathrm{SL}(2,\mathbb{R})$ and $\mathbb{R}^2$.

Remark. In n dimensions, the analogues of equations (1) and (2) are valid, but $M(t)$ must belong to the group of $2n\times 2n$ symplectic matrices, $\mathrm{Sp}(n,\mathbb{R})$. This follows from the fact that canonical transformations preserve the symplectic structure on phase space.

3. The time-periodic case

In this section we will prove theorem 1 in the one-dimensional case. To do this, we will first classify the various affine transformations that occur in equation (2). Then we will identify the spectral types of the quantum analogues of the elements of each of the classes of affine transformations. Since every Floquet operator associated with quadratic Hamiltonians is one of these operators (up to a phase), theorem 1 will follow from this.

We begin with some notation. For quantum mechanical operators A and B, we write $A \sim B$ if $A = \exp(i\phi)B$ for some real ϕ. We write $A \simeq B$ if A is unitarily equivalent to B. We denote the unitary implementation of the translation's $T\left[\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\right]$ by $W\left[\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\right]$. These operators are only determined up to phases, but here we choose to specify them by the formula

$$W\left[\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\right] = \exp[i(\beta q - \alpha p)].$$

We denote the unitary implementation of the action of $M \in SL(2, \mathbb{R})$ by $R[M]$. Again, these operators are only determined up to phases, so we will assume they have been chosen in some arbitrary way.

As mentioned at the end of § 2, the relevant group of affine transformations of $\mathbb{R}^2$ is the semi-direct product of $SL(2, \mathbb{R})$ and $\mathbb{R}^2$. Thus, we can unitarily implement the affine group on $\mathbb{R}^2$ by the products $W\left[\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\right] R[M]$ modulo phases. The fact that we are representing a semi-direct product follows from the relation

$$\begin{aligned} R[M]\, W\left[\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\right] R[M]^{-1} &\sim W\left[JM^{t}J^{-1}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\right] \\ &\sim W\left[M^{-1}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\right] \end{aligned} \tag{4}$$

where J is the matrix $\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$. One should note that this is not the same semi-direct product described by equation (3). Equation (3) describes the action of the affine group on the position q and the momentum p. Equation (4) describes the action on quantum mechanical state vectors.

We wish to find the spectral type of all operators of the form $W\left[\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\right] R[M]$. To do this, we will first classify all such operators up to unitary equivalence. As a first step in this classification, we note that if $N = AMA^{-1}$ with M, $A \in SL(2, \mathbb{R})$, then

$$\begin{aligned} W\left[\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\right] R[N] &\sim W\left[\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\right] R[A]\, R[M]\, R[A]^{-1} \\ &\sim R[A]\, W\left[A\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\right] R[M]\, R[A]^{-1}. \end{aligned}$$

Therefore

$$W\left[\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\right] R[N] \simeq W\left[A\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\right] R[M].$$

Thus, to list the qualitative spectral properties of the collection of operators of the form $W\left[\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\right] R[M]$, one only needs to consider these operators for one M in each conjugacy class and all α and β.

To describe the conjugacy classes, we first note that there are three types of matrices in $\mathrm{SL}(2,\mathbb{R})$. They are called elliptic, hyperbolic and parabolic, and they are classified as follows.

Elliptic—those matrices in $\mathrm{SL}(2,\mathbb{R})$ with two distinct complex eigenvalues (on the unit circle) and plus or minus the identity.

Hyperbolic—those matrices in $\mathrm{SL}(2,\mathbb{R})$ with two distinct real eigenvalues.

Parabolic—those non-diagonalisable matrices in $\mathrm{SL}(2,\mathbb{R})$ with 1 or -1 as a double eigenvalue.

Since elements of $\mathrm{SL}(2,\mathbb{R})$ have determinant 1, those matrices $M\in\mathrm{SL}(2,\mathbb{R})$ with $|\mathrm{trace}\, M|<2$ are elliptic. Those with $|\mathrm{trace}\, M|>2$ are hyperbolic. Those with trace ± 2 are elliptic if they are diagonalisable and parabolic if they are not.

Every elliptic matrix M in $\mathrm{SL}(2,\mathbb{R})$ is conjugate to the rotation matrix

$$E(\varphi)=\begin{pmatrix}\cos\varphi & \sin\varphi\\ -\sin\varphi & \cos\varphi\end{pmatrix}$$

for some $\varphi\in[0,2\pi]$. To see this, we first note that it is true for the cases of plus or minus the identity matrix. Next, we assume M is elliptic, but not plus or minus the identity. Let $\exp(\mathrm{i}\varphi)$ and $\exp(-\mathrm{i}\varphi)$ be the eigenvalues of M, and let v and $\bar{v}$ be the corresponding eigenvectors. We may assume that v and φ have been chosen so that the matrix A, whose columns are $(v+\bar{v})/2$ and $(v-\bar{v})/2\mathrm{i}$, belongs to $\mathrm{SL}(2,\mathbb{R})$. Then $A^{-1}MA=E(\varphi)$.

Every hyperbolic matrix M in $\mathrm{SL}(2,\mathbb{R})$ is conjugate to one of the matrices

$$H_\pm(\varphi)=\begin{pmatrix}\pm\exp(\varphi) & 0\\ 0 & \pm\exp(-\varphi)\end{pmatrix}$$

for some non-zero $\varphi\in\mathbb{R}$. To see this, choose independent eigenvectors v_1 and v_2 for M so that the matrix A, whose columns are v_1 and v_2, belongs to $\mathrm{SL}(2,\mathbb{R})$. Let $\pm\exp(\varphi)$ be the eigenvalue of M corresponding to v_1. Then $\pm\exp(-\varphi)$ is the eigenvalue corresponding to v_2, and $A^{-1}MA=H_\pm(\varphi)$.

Every parabolic matrix M in $\mathrm{SL}(2,\mathbb{R})$ is conjugate to one of the matrices

$$P_\pm(\varphi)=\begin{pmatrix}\pm 1 & \varphi\\ 0 & \pm 1\end{pmatrix}$$

where $\varphi\in\mathbb{R}\backslash\{0\}$. To prove this, let v_1 be an eigenvector of M with eigenvalue ± 1. Choose v_2 so that $(M\mp I)v_2=\varphi v_1$, where φ has been chosen so that the matrix A, with columns v_1 and v_2, belongs to $\mathrm{SL}(2,\mathbb{R})$. Then $A^{-1}MA=P_\pm(\varphi)$. Furthermore, by conjugating $P_\pm(\varphi)$ with a diagonal element of $\mathrm{SL}(2,\mathbb{R})$, one obtains $P_\pm(\varphi')$ for some φ'. The only restriction on φ' is that it must have the same sign as φ.

Thus, the set of matrices $\{E(\varphi),\, H_\pm(\varphi),\, P_\pm(\varphi)\}$ contains representatives of all of the conjugacy classes of $\mathrm{SL}(2,\mathbb{R})$. With this information, we can now describe the spectra of all operators fo the form $R[M]=W\left[\begin{pmatrix}0\\ 0\end{pmatrix}\right] R[M]$. After studying this special case, we will return to the general case of operators of the form $W\left[\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\right] R[M]$.

The mappings $\varphi \to E(\varphi)$, $\varphi \to H_+(\varphi)$ and $\varphi \to P_+(\varphi)$ are one parameter groups whose generators are

$$\mathbb{E} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \mathbb{H} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \mathbb{P} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

We will now show that the corresponding generators in quantum mechanics are $(p^2+q^2)/2$, $(q\cdot p+p\cdot q)/2$, and $p^2/2$. To do this, we first claim that $R[E(\varphi)] \sim U(\varphi)$, where

$$U(\varphi) = \exp[\mathrm{i}\tfrac{1}{2}\varphi(p^2+q^2)].$$

To see this, notice that

$$\begin{pmatrix} q(\varphi) \\ p(\varphi) \end{pmatrix} = E(\varphi) \begin{pmatrix} q \\ p \end{pmatrix}$$

satisfies the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}\varphi} \begin{pmatrix} q(\varphi) \\ p(\varphi) \end{pmatrix} = \mathbb{E} \begin{pmatrix} q(\varphi) \\ p(\varphi) \end{pmatrix}$$

in $\mathbb{R}^2$, with the initial condition

$$\begin{pmatrix} q(0) \\ p(0) \end{pmatrix} = \begin{pmatrix} q \\ p \end{pmatrix}.$$

In the quantum mechanics,

$$\begin{pmatrix} q(\varphi) \\ p(\varphi) \end{pmatrix} = U(\varphi) \begin{pmatrix} q \\ p \end{pmatrix} U(\varphi)^{-1}$$

satisfies the same initial condition and the same differential equation, that is

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\varphi} \begin{pmatrix} q(\varphi) \\ p(\varphi) \end{pmatrix} &= \tfrac{1}{2}\mathrm{i}\, U(\varphi) \begin{pmatrix} [p^2+q^2, q] \\ [p^2+q^2, p] \end{pmatrix} U(\varphi)^{-1} \\ &= \tfrac{1}{2}\mathrm{i}\, U(\varphi) \begin{pmatrix} -2\mathrm{i}p \\ 2\mathrm{i}q \end{pmatrix} U(\varphi)^{-1} \\ &= \mathbb{E} \begin{pmatrix} q(\varphi) \\ p(\varphi) \end{pmatrix}. \end{aligned}$$

Thus, $U(\varphi)$ implements $E(\varphi)$, and the claim is proved.

Similar arguments show that

$$R[H_+(\varphi)] \sim \exp[\tfrac{1}{2}\mathrm{i}\varphi(p\cdot q+q\cdot p)]$$

and

$$R[P_+(\varphi)] \sim \exp(\tfrac{1}{2}\mathrm{i}\varphi p^2).$$

The spectra of the self-adjoint operators p^2+q^2, $p\cdot q+q\cdot p$ and p^2 are well known to be purely discrete, transient absolutely continuous (see e.g. Perry 1984, proposition 6.2) and transient absolutely continuous, respectively. Thus, $R[E(\varphi)]$, $R[H_+(\varphi)]$ and $R[P_+(\varphi)]$ have strictly pure point, transient absolutely continuous and transient absolutely continuous, respectively.

The spectra of $R[H_-(\varphi)]$ and $R[P_-(\varphi)]$ are also purely transient absolutely continuous. Using arguments similar to those above, we see that

$$R[H_-(\varphi)] \sim \Gamma R[H_+(\varphi)]$$

and

$$R[P_-(\varphi)] \sim \Gamma R[P_+(-\varphi)]$$

where $\Gamma: L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is the reflection which takes the wavefunction $\Psi(x)$ into $\Psi(-x)$. The operator $\Gamma R[H_+(\varphi)]$ is diagonalised by the generalised eigenfunctions $|x|^{i\lambda}$ and $|x|^{i\lambda}\,\mathrm{sgn}(x)$. The operator $\Gamma R[P_+(-\varphi)]$ is diagonalised by the eigenfunctions $\sin(\lambda^{1/2}x)$ and $\cos(\lambda^{1/2}x)$. Using the expansions in these eigenfunctions, one can easily see that the spectra of $R[H_-(\varphi)]$ and $R[P_-(\varphi)]$ are purely transient absolutely continuous.

This completes the classification of the spectra of the operators $R[M]$ for $M \in \mathrm{SL}(2, \mathbb{R})$.

Remark. Consider the 'parametric resonance problem' (Arnold 1978) $H(t) = p^2 + [\lambda(t) + C]q^2$, with $\lambda(t)$ periodic but not constant. The matrix $M(T)$ depends on C in an interesting way. As C is increased from $-\infty$, $M(T)$ starts out hyperbolic. At some critical value of C, it becomes parabolic. As C is increased further, there is an interval in which $M(T)$ is elliptic; this band is open. At the end of the band there is a point at which $M(T)$ is parabolic. Above this point, there is a gap in which $M(T)$ is again hyperbolic. And so on. For generic periodic functions $\lambda(t)$, there are infinitely many elliptic bands and hyperbolic gaps separated by parabolic points. The proof of this is the standard Floquet analysis for a periodic Schrödinger equation (Hill's equation).

In the quantum mechanical analogue of this problem, the unitary Floquet operator has spectral type which switches back and forth between strictly pure point and strictly transient absolutely continuous as C is varied.

We now turn to the problem of classifying the spectra of all operators of the form $W\left[\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\right] R[M]$, with $M \in \mathrm{SL}(2, \mathbb{R})$ and $\begin{pmatrix}\alpha\\ \beta\end{pmatrix} \neq \begin{pmatrix}0\\ 0\end{pmatrix}$. As before, we need only consider the situations in which M is one of the operators $E(\varphi)$, $H_\pm(\varphi)$ or $P_\pm(\varphi)$.

The argument which we present below is not our original argument. We wish to thank Barry Simon for the proof which we present, since it is more enlightening and technically less complicated.

We separately study three cases.

Case 1. $M = E(\varphi)$ with φ an integral multiple of 2π.
In this (trivial) case, M is the identity operator, and $R[M]$ is simply multiplication by a phase factor. Therefore, $W\left[\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\right] R[M] \sim W\left[\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\right]$.

For some angle ω,

$$W\left[\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\right] \sim R[E(\omega)]\, W\left[\begin{pmatrix}\gamma\\ 0\end{pmatrix}\right] R[E(\omega)]^{-1},$$

where $W\left[\begin{pmatrix}\gamma\\ 0\end{pmatrix}\right]$ is a non-trivial translation. Therefore, $W\left[\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\right] R[M]$ has a purely transient absolutely continuous spectrum.

Case 2. M is $H_{\pm}(\varphi)$ for $\varphi \neq 0$, $P_{-}(\varphi)$ for $\varphi \neq 0$, or $E(\varphi)$ with φ not an integral multiple of 2π.

In this case, we will show that $W\left[\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\right] R[M]$ is unitarily equivalent to $R[M]$ modulo phases. The equivalence arises from a translation of the origin in the classical phase space.

We first note that the number one does not belong to the spectrum of M. Thus, given any $\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$, we can set

$$\begin{pmatrix}\gamma\\ \delta\end{pmatrix} = (1-M)^{-1}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}.$$

The affine map of $\mathbb{R}^2$ which corresponds to $W\left[\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\right] R[M]$ is $T\left[\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\right] M$. This map takes the point $\begin{pmatrix}q\\ p\end{pmatrix}$ into the point $M\begin{pmatrix}q\\ p\end{pmatrix}+\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$. By a simple calculation, this is the same as the mapping $T\left[\begin{pmatrix}\gamma\\ \delta\end{pmatrix}\right] M\, T\left[\begin{pmatrix}\gamma\\ \delta\end{pmatrix}\right]^{-1}$, which maps the point $\begin{pmatrix}q\\ p\end{pmatrix}$ into the point $M\begin{pmatrix}q-\gamma\\ p-\delta\end{pmatrix}+\begin{pmatrix}\gamma\\ \delta\end{pmatrix}$.

The corresponding quantum mechanical operator, $W\left[\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\right] R[M]$, is thus equal to $W\left[\begin{pmatrix}\gamma\\ \delta\end{pmatrix}\right] R[M]\, W\left[\begin{pmatrix}\gamma\\ \delta\end{pmatrix}\right]^{-1}$ modulo phases. Since this operator is unitarily equivalent to $R[M]$, the spectral type of $W\left[\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\right] R[M]$ is the same as that of $R[M]$, which we classified earlier.

Case 3. $M = P_{+}(\varphi)$.
The quantum transformation $W\left[\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\right] R[P_{+}(\varphi)]$ corresponds to the classical affine transformation $T\left[\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\right] P_{+}(\varphi)$, which takes the point $\begin{pmatrix}q\\ p\end{pmatrix}$ into the point

$$P_{+}(\varphi)\begin{pmatrix}q\\ p\end{pmatrix}+\begin{pmatrix}\alpha\\ \beta\end{pmatrix}=\begin{pmatrix}q+\varphi p+\alpha\\ p+\beta\end{pmatrix}.$$

This transformation can be rewritten as the mapping which takes the point $\begin{pmatrix}q\\ p\end{pmatrix}$ into the point $P_{+}(\varphi)\begin{pmatrix}q\\ p+\gamma\end{pmatrix}+\begin{pmatrix}\beta\varphi/2\\ \beta-\gamma\end{pmatrix}$, where $\gamma=\alpha/\varphi-\beta/2$. This is the transformation

$$T\left[\begin{pmatrix}0\\ -\gamma\end{pmatrix}\right] T\left[\begin{pmatrix}\beta\varphi/2\\ \beta\end{pmatrix}\right] P_{+}(\varphi)\, T\left[\begin{pmatrix}0\\ \gamma\end{pmatrix}\right],$$

which is obviously conjugate to $T\left[\begin{pmatrix}\beta\varphi/2\\ \beta\end{pmatrix}\right] P_{+}(\varphi)$.

By a simple calculation, $T\left[\begin{pmatrix}\beta\varphi/2\\ \beta\end{pmatrix}\right] P_+(\varphi)$ coincides with the classical phase flow at time $t=\varphi$ for the Hamiltonian $H=p^2/2+\beta q/\varphi$. Therefore, modulo phases and unitary equivalence, $W\left[\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\right] R[P_+(\varphi)]$ is the Stark effect propagator $\exp[-i\varphi(p^2/2+\beta q/\varphi)]$. If $\beta=0$, this is a free propagator, whose spectrum is purely transient absolutely continuous. If $\beta\neq 0$, it is the propagator for a Stark Hamiltonian with a non-zero field. Such Hamiltonians are unitarily equivalent to multiplication by q (Reed and Simon 1978, p 118) and hence have purely transient absolutely continuous spectra.

This concludes the proof of theorem 1 in one dimension. In higher dimensions, one must do similar analyses with $SL(2,\mathbb{R})$ replaced by the symplectic groups.

Remark. If one studies the standard forced harmonic oscillator, $H(t)=\frac{1}{2}\{p^2+q^2\}+q\cos(\omega t)$, then one sees some of the situations described above. If $\omega\neq 1$, the Floquet operator has a strictly discrete spectrum. If $\omega=1$, the spectrum is purely transient absolutely continuous.

4. The proof of theorem 2

In this section we consider the simple model with random time dependence that was introduced in the introduction. We will first study the classical motion, and then study the quantum evolution.

We will be primarily interested in the classical phase flow at integer times. We define the matrix $M_{\omega_k}\in SL(2,\mathbb{R})$ by the relation

$$\begin{pmatrix}q(k)\\ p(k)\end{pmatrix}=M_{\omega_k}\begin{pmatrix}q(k-1)\\ p(k-1)\end{pmatrix}.$$

The assumption on the probability distribution μ is that the subgroup G of $SL(2,\mathbb{R})$ generated by the matrices M_{ω_k} contains at least two elements which have no common eigenvector. This assumption implies (Ishii and Matsuda 1970, Ishii 1973) that the hypotheses of Fürstenberg's theorem on products of random matrices (Fürstenberg 1963) are fulfilled. From this theorem we can conclude that for almost all vectors $\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$, we have

$$\lim_{n\to\infty}\left(\frac{1}{n}\log\left\|M_{\omega_n}M_{\omega_{n-1}}\dots M_{\omega_1}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\right\|=\delta\right)>0 \tag{5}$$

for almost all choices of the ω_k.

We now turn to the quantum mechanical analogue. Let Ψ be a Schwartz function. Then

$$\begin{aligned}|\langle\Psi, U(t,0)\Psi\rangle| &\leq \|(1+q^2+p^2)^{1/2}\Psi\|\,\|(1+q^2+p^2)^{-1/2}U(t,0)\Psi\|\\ &\leq C_1\|(1+q^2+p^2)^{-1/2}U(t,0)\Psi\|.\end{aligned}$$

To prove the theorem, it is clearly sufficient to prove that the norm in the last expression tends to zero exponentially as t tends to infinity. To show this, let $M(t)$ be the classical phase flow propagator that corresponds to $U(t,0)$ and let $\alpha(t)$ be its norm. By using the polar decomposition, we see that $M(t)=|M(t)|\,V(t)$, where $|M(t)|$ is real symmetric and $V(t)$ is orthogonal. By diagonalising $|M(t)|$, we see that $M(t)=A(t)B(t)C(t)$, where $A(t)$ and $B(t)$ are orthogonal and $B(t)$ is the matrix $\begin{bmatrix}\alpha(t) & 0\\ 0 & \alpha(t)^{-1}\end{bmatrix}$.

Now, $U(t,0)\sim R[A(t)]\,R[B(t)]\,R[C(t)]$, so

$$\|(1+q^2+p^2)^{-1/2}U(t,0)\Psi\| = \|(1+\alpha(t)^2q^2+\alpha(t)^{-2}p^2)^{-1/2}R[C(t)]\Psi\|$$
$$\leqslant \|(1+\alpha(t)^2q^2)^{-1/2}R[C(t)]\Psi\|$$

Since $R[C(t)]\sim\exp[-i\beta(t)(p^2+q^2)]$ for some function $\beta(t)$, the set $\{R[C(t)]\Psi: 0\leqslant t<\infty\}$ is contained in a compact subset of Schwartz space. Thus, $\|R[C(t)]\Psi\|_\infty\leqslant C_2$, and so,

$$\|(1+\alpha(t)^2q^2)^{-1/2}R[C(t)]\Psi\| \leqslant \|(1+\alpha(t)^2q^2)^{-1/2}\|_2\,\|R[C(t)]\Psi\|_\infty$$
$$\leqslant C_3\alpha(t)^{-1}.$$

Since (5) implies that $\alpha(t)$ grows exponentially, this estimate implies the theorem.

Acknowledgments

It is a pleasure to thank Jean Bellissard for several useful conversations. It is also a pleasure to thank Barry Simon for pointing out errors in the original manuscript, and for suggesting several simplifications of our original proofs.

The authors are supported in part by the National Science Foundation, GAH under grant number MCS-8301277, ML under PHY-8116101A02 and JS under MCS-8301709.

Note added in proof. Certain ideas which overlap with those of this paper can be found in A M Perelomov (1977 *Sov. Phys.-Usp.* **20** 703-21).

References

Arnold V I 1978 *Mathematical Methods of Classical Mechanics* (New York: Springer)
Avron J E and Simon B 1981 *J. Funct. Anal.* **43** 1-31
Fürstenberg H 1963 *Trans. Am. Math. Soc.* **108** 3-28
Goldstein H 1965 *Classical Mechanics* (Reading, MA: Addison-Wesley)
Hagedorn G A 1981 *Ann. Phys., NY* **135** 58-70
—— 1985 *Ann. Inst. H Poincaré* **42** 363-74
Ishii K 1973 *Prog. Theor. Phys. Suppl.* **53** 77-138
Ishii K and Matsuda H 1970 *Prog. Theor. Phys. Suppl.* **45** 56-89
Perry P A 1984 *Scattering Theory by the Enss Method* (New York: Harwood Academic)
Reed M and Simon B 1978 *Methods of Modern Mathematical Physics IV, Analysis of Operators* (New York: Academic)